On Multi-parametric Nonlinear Programming and Explicit Nonlinear Model Predictive Control

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Abstract

A numerical algorithm for approximate multi-parametric nonlinear programming is developed. It allows approximate solutions to nonlinear optimization problems to be computed as explicit piecewise linear functions of the problem parameters. In control applications such as nonlinear constrained model predictive control this allows efficient online implementation in terms of an explicit piecewise linear state feedback without any real-time optimization.

1 Introduction

Exact solutions to multi-parametric quadratic and linear programs (mp-QP/mp-LP) can be found using the methods of e.g. [1, 2, 3]. These recently developed algorithms allows the off-line computation of explicit piecewise linear (PWL) state feedback control laws for linearly constrained linear/quadratic optimal control problems. This facilitates efficient real-time implementation of constrained optimal feedback control strategies such as model predictive control (MPC) without the use of real-time optimization, see also [4, 5, 6]. MPC can be implemented on low-cost hardware and with low software complexity in embedded systems. This opens new application areas for MPC, which has traditionally been restricted to slow process plants.

For general multi-parametric nonlinear programs (mp-NLPs) one cannot expect to find exact solutions. There is a large body of theory that develops local regularity conditions and local sensitivity results [7, 8], and algorithms for large parameter variations are derived for single-parametric problems [9]. Here we propose an approximate mp-NLP algorithm utilizing NLP and mp-QP algorithms to solve local sub-problems, with applications to nonlinear constrained MPC problems in mind. Approximate mp-QP methods with application to linear constrained MPC problems have been suggested by [10, 11, 12]. Related function approximation methods for non-linear optimal control are described in [13, 14, 15].

2 Problem formulation

Consider the nonlinear dynamic optimization problem

$$J(u[0,T],x[0,T]) \triangleq \int_0^T l(x(t),u(t),t)dt + S(x(T),T) \qquad (1)$$

subject to the inequality constraints for $t \in [0,T]$

$$u_{min} \le u(t) \le u_{max} \qquad (2)$$
$$g(x(t),u(t)) \le 0 \qquad (3)$$

and the ordinary differential equation (ODE) given by

$$\frac{d}{dt}x(t) = f(x(t),u(t)) \qquad (4)$$

with given initial condition $x(0) \in X \subset \mathbb{R}^n$. The input signal $u[0,T]$ is assumed to be piecewise constant and parameterized by a vector $U \in \mathbb{R}^p$ such that $u(t) = \mu(t,U) \in \mathbb{R}^r$ is piecewise continuous. The solution to (4) is assumed in the form $x(t) = \phi(t,U,x(0))$ for $t \in [0,T]$ and some piecewise continuous function ϕ. Relaxing the inequality constraints (3) to hold only at N time instants $\{t_1,t_2,...,t_N\} \subset [0,T]$, we can rewrite the optimization problem in the following standard parametric form (direct single shooting, e.g. [16]) where the ODE constraint (4) has been eliminated by substituting its solution ϕ into the cost and constraints; minimize with respect to U the cost

$$V(U;x(0)) \triangleq \int_0^T l(\phi(t,U,x(0)),\mu(t,U),t)dt + S(\phi(T,U,x(0)),T) \qquad (5)$$

subject to

$$G(U;x(0)) \triangleq \begin{pmatrix} \tilde{G}(U;x(0)) \\ U - U_{max} \\ U_{min} - U \end{pmatrix} \le 0 \qquad (6)$$

with blocks $\tilde{G}_i(U;x(0)) \triangleq g(\phi(t_i,U,x(0)),\mu(t_i,U))$. Eqs. (5) - (6) define an mp-NLP, since it is an NLP in U parameterized by the initial state vector $x(0)$. We note that the introduction of common modifications such as terminal constraints and infeasibility relaxations still gives an mp-NLP. Assume the solution exists, and let it be denoted $U^*(x(0))$. In the special case when V and G are quadratic and linear, respectively, in both U and $x(0)$, a solution can be found explicitly and exactly as a continuous PWL mapping $U^*(x(0))$ using mp-QP [4, 3].

Here we suggest to utilize an mp-QP algorithm to approximately solve the mp-NLP (5)-(6). In the mp-QP case, this algorithm will iteratively build a polyhedral partition of the state-space with an exact solution corresponding to a fixed active set within each polyhedral critical region. This leads to a PWL solution $U^*(x)$ since a fixed active set leads to a solution that is linear in x, [2]. In the mp-NLP case we keep the PWL structure of the solution, but in each polyhedral region we approximate the (exact) nonlinear solution by a PWL approximate solution found by solving a mp-QP constructed as a locally accurate quadratic approximation to V and linear approximation to G. Under regularity assumptions on V and G, one may expect that the approximation error and constraint violations will be small if each of the regions are sufficiently small. We therefore suggest to analyze the approximation error within each region and introduce a sub-partitioning of some regions when needed in order to keep the approximation error and constraint violations within specified bounds.

3 Properties of the mp-NLP

For a given $x_0 \in X$ the well known Karush-Kuhn-Tucker (KKT) first-order conditions [17]

$$
\begin{aligned}
\nabla_U L(U_0; x_0) &= 0 && (7)\\
\mathrm{diag}(\lambda_0) G(U_0; x_0) &= 0 && (8)\\
\lambda_0 &\geq 0 && (9)\\
G(U_0; x_0) &\leq 0 && (10)
\end{aligned}
$$

are necessary for a local minimum U_0, with associated Lagrange multiplier λ_0 and the Lagrangian defined as

$$
L(U, \lambda; x) \triangleq V(U; x) + \lambda^T G(U; x) \quad (11)
$$

Consider the optimal active set $\mathcal{A}_0$ at x_0, i.e. a set of indices to active constraints in (10). The above conditions are sufficient provided the following second order condition holds:

$$
z^T \nabla^2_{UU} L(U_0, \lambda_0; x_0) z > 0, \ \text{ for all } z \in \mathcal{F} - \{0\} \quad (12)
$$

with $\mathcal{F}$ being the set of all directions where it is not clear from first order conditions if the cost will increase or decrease:

$$
\begin{aligned}
\mathcal{F} = \; & \{ z \in \mathbb{R}^p \mid \nabla_U G_{\mathcal{A}_0}(U_0; x_0) z \geq 0, \\
& \nabla_U G_i(U_0; x_0) z = 0, \text{ for all } i \text{ with } (\lambda_0)_i > 0 \} \quad (13)
\end{aligned}
$$

The notation $G_{\mathcal{A}_0}$ means the rows of G with indices in $\mathcal{A}_0$. The following result gives local regularity conditions for the optimal solution, Lagrange multipliers and optimal cost as functions of x.

Assumption A1. V and G are twice continuously differentiable in a neighborhood of (U_0, x_0).

Assumption A2. The sufficient conditions (7)-(10) and (12) for a local minimum at U_0 hold.

Assumption A3. Linear independence constraint qualification (LICQ) holds, i.e. the active constraint gradients $\nabla_U G_{\mathcal{A}_0}(U_0; x_0)$ are linearly independent.

Assumption A4. Strict complementary slackness holds, i.e. $(\lambda_0)_{\mathcal{A}_0} > 0$.

Theorem 1 *Consider the problem (5) - (6), and let $x_0 \in X$ and U_0 be given. If A1 - A3 holds, then*

1. *U_0 is a local isolated minimum.*

2. *For x in a neighborhood of x_0, there exists a unique continuous function $U^*(x)$ satisfying $U^*(x_0) = U_0$ and the sufficient conditions for a local minimum.*

3. *Assume in addition A4 holds, and let x be in a neighborhood of x_0. Then $U^*(x)$ is differentiable and the associated Lagrange multipliers $\lambda^*(x)$ exists, and are unique and continuously differentiable. Finally, the set of active constraints is unchanged, and the active constraint gradients are linearly independent at $U^*(x)$.*

□

Parts 1 and 2 are due to [18], while part 3 is due to Theorem 3.2.2 in [7]. Related results for slightly different conditions, and extensions that show the existence and computation of directional derivatives of the solution with respect to x at x_0 can be found in [7, 8, 19] and others. For the fixed active set $\mathcal{A}_0$ the KKT conditions (7)-(8) reduces to the following system of equations parameterized by x:

$$
\nabla_U V(U(x); x) + \sum_{i \in \mathcal{A}_0} \lambda_i(x) \nabla_U G_i(U(x); x) = 0 \quad (14)
$$

$$
G_{\mathcal{A}_0}(U(x); x) = 0 \quad (15)
$$

The functions $U(x)$ and $\lambda(x)$ implicitly defined by (14)-(15) are optimal only for those x where the active set $\mathcal{A}_0$ is optimal. Assuming λ and U are well defined on X, we characterize the critical region $\mathcal{X}_{\mathcal{A}_0}$ where the solution corresponding to the fixed active set $\mathcal{A}_0$ is optimal:

$$
\mathcal{X}_{\mathcal{A}_0} \triangleq \{ x \in X \mid \lambda(x) \geq 0, \; G(U(x); x) \leq 0 \} \quad (16)
$$

There is a finite number of candidate active sets, so this result suggests a finite partition of X with a piecewise solution to the mp-NLP, similar to [2, 3] for mp-QPs. Although explicit exact solutions cannot be found in the general nonlinear case, the above result indicates that it is meaningful to search for a continuous approximation to the optimal solution as a function of x.

4 Local mp-QP approximation to mp-NLP

In this section we study how the cost function and constraints can be locally approximated by mp-QP problems. Let $x_0 \in X$ be arbitrary and denote the corresponding optimal solution $U_0 = U^*(x_0)$. Taylor series expansions of V and G about the point (U_0, x_0) leads to the following locally approximate mp-QP problem:

$$
\begin{aligned}
V_0(U; x) \triangleq \; & \frac{1}{2}(U - U_0)^T H_0 (U - U_0) && (17)\\
& + (D_0 + F_0(x - x_0))(U - U_0) + Y_0(x; x_0)
\end{aligned}
$$

subject to

$$
G_0(U - U_0) \leq E_0(x - x_0) + T_0 \quad (18)
$$

The cost and constraints are defined by the matrices

$$
H_0 \triangleq \nabla^2_{UU} V(U_0; x_0), \quad F_0 \triangleq \nabla^2_{xU} V(U_0; x_0)
$$

$$
D_0 \triangleq \nabla_U V(U_0; x_0), \quad G_0 \triangleq \begin{pmatrix} \nabla_U \tilde{G}(U_0; x_0) \\ I \\ -I \end{pmatrix}
$$

$$
E_0 \triangleq \begin{pmatrix} -\nabla_x \tilde{G}(U_0; x_0) \\ 0 \\ 0 \end{pmatrix}, \quad T_0 \triangleq \begin{pmatrix} -\tilde{G}(U_0; x_0) \\ U_{max} - U_0 \\ U_0 - U_{min} \end{pmatrix}
$$

$$
\begin{aligned}
Y_0(x; x_0) \triangleq \; & V(U_0; x_0) + \nabla_x V(U_0; x_0)(x - x_0) \\
& + \tfrac{1}{2}(x - x_0)^T \nabla^2_{xx} V(U_0; x_0)(x - x_0)
\end{aligned}
$$

Let the PWL solution to the mp-QP (17) - (18) be denoted $U_{QP}(x)$ with associated Lagrange multipliers $\lambda_{QP}(x)$. This solution satisfies the following KKT conditions

$$
\begin{aligned}
H_0 \left(U_{QP}(x) - U_0 \right) + F_0(x - x_0) & \\
+ D_0 + G_0^T \lambda_{QP}(x) &= 0 && (19)\\
\mathrm{diag}(\lambda_{QP}(x)) \left(G_0(U_{QP}(x) - U_0) \right. & \\
\left. - E_0(x - x_0) - T_0 \right) &= 0 && (20)\\
\lambda_{QP}(x) &\geq 0 && (21)\\
G_0 \left(U_{QP}(x) - U_0 \right) - E_0(x - x_0) - T_0 &\leq 0 && (22)
\end{aligned}
$$

Consider the optimal active set $\mathcal{A}$ of the QP (17) - (18) at a given $x \in X$, and let $G_{0,\mathcal{A}}$ and $\lambda_{QP,\mathcal{A}}$ denote the rows of G_0 and λ_{QP}, respectively, with indices in $\mathcal{A}$. Eqs. (19) - (20) define the following linear equations

$$\begin{pmatrix} H_0 & G_{0,\mathcal{A}}^T \\ G_{0,\mathcal{A}} & 0 \end{pmatrix} \begin{pmatrix} U_{QP,\mathcal{A}}(x) - U_0 \\ \lambda_{QP,\mathcal{A}}(x) \end{pmatrix} = \begin{pmatrix} F_0(x - x_0) + D_0 \\ E_0(x - x_0) + T_0 \end{pmatrix} \quad (23)$$

The following results is an extension of Theorem 2 in [4] (where $H_0 > 0$ was assumed in addition to LICQ).

Assumption A5. For an optimal active set $\mathcal{A}$, the matrix $G_{0,\mathcal{A}}$ has full row rank (LICQ) and $Z_{0,\mathcal{A}}^T H_0 Z_{0,\mathcal{A}} > 0$, where the columns of $Z_{0,\mathcal{A}}$ is a basis for null$(G_{0,\mathcal{A}})$.

Theorem 2 *Consider the problem (17)-(18), and let X be a polyhedral set with $x_0 \in X$. If assumption A5 holds, then the system of linear equations (23) has a unique solution and the critical region where the solution is optimal is given by the polyhedral set*

$$\begin{aligned} \mathcal{X}_{0,\mathcal{A}} \triangleq \{ & x \in X \mid \lambda_{QP,\mathcal{A}}(x) \geq 0, \\ & G_0(U_{QP,\mathcal{A}}(x) - U_0) \leq E_0(x - x_0) + T_0 \} \end{aligned}$$

Hence, $U_{QP}(x) = U_{QP,\mathcal{A}}(x)$ and $\lambda_{QP}(x) = \lambda_{QP,\mathcal{A}}(x)$ if $x \in \mathcal{X}_{0,\mathcal{A}}$, and the solution U_{QP} is a continuous, PWL function of x defined on a polyhedral partition of X.

Proof. Non-singularity of the matrix on the left-hand-side of (23) follows from standard 2nd order considerations such as Lemma 16.1 in [17], due to Assumption A5. The rest of the proof is similar to [4].

□

Algorithms for solving such an mp-QP (with straightforward modifications to account for the relaxed second-order condition A5) are given in [3, 4]. The following result compares the primal and dual local QP solution with the global NLP solution.

Theorem 3 *Consider the problem (5)-(6). Let $x_0 \in X$ and suppose there exists a U_0 satisfying assumptions A1 - A4. Then for x in a neighbourhood of x_0*

$$U_{QP}(x) - U^*(x) = \mathcal{O}(||x - x_0||_2^2) \quad (24)$$

$$\lambda_{QP}(x) - \lambda^*(x) = \mathcal{O}(||x - x_0||_2^2) \quad (25)$$

Proof. Let the neighborhood of x_0 under consideration be restricted to $\mathcal{X}_{0,\mathcal{A}_0}$, where $\mathcal{A}_0$ is the optimal active set at x_0. The first KKT condition for the QP is

$$\begin{aligned} H_0 \left(U_{QP}(x) - U_0 \right) + F_0(x - x_0) & \\ + \left(D_0 + G_0^T \lambda_{QP}(x) \right) &= 0 \end{aligned} \quad (26)$$

Since $U_0 = U^*(x_0)$ we have $U^*(x) - U_0 = \mathcal{O}(||x - x_0||_2)$, and the first KKT condition (7) for the NLP can be rewritten as follows using a Taylor series expansion

$$\begin{aligned} 0 &= \nabla_U V(U^*(x); x) + \nabla_U^T G(U^*(x); x) \lambda^*(x) \quad (27) \\ &= D_0 + H_0(U^*(x) - U_0) + F_0(x - x_0) + G_0^T \lambda_{QP}(x) \\ &\quad + G_0^T(\lambda^*(x) - \lambda_{QP}(x)) + \mathcal{O}(||x - x_0||_2^2) \\ &\quad + \mathcal{O}(||x - x_0||_2)(\lambda^*(x) - \lambda_{QP}(x)) \quad (28) \end{aligned}$$

Comparing (26) and (28) we get

$$H_0 \left(U_{QP}(x) - U^*(x) \right) + G_0^T \left(\lambda_{QP}(x) - \lambda^*(x) \right) = \mathcal{O}(||x - x_0||_2^2) \quad (29)$$

From Theorem 1, part 3, it is known that the set of active constraints is unchanged in a neighbourhood of x_0. Hence, for the QP we have

$$G_0 \left(U_{QP}(x) - U_0 \right) = E_0(x - x_0) + T_0 \quad (30)$$

When x is in a neighbourhood of x_0, Taylor expanding the NLP constraints gives

$$\begin{aligned} 0 &= G(U^*(x); x) \quad (31) \\ &= G_0(U^*(x) - U_0) - E_0(x - x_0) - T_0 + \mathcal{O}(||x - x_0||_2^2) \end{aligned}$$

Comparing (30) and (31) it follows that

$$G_0 \left(U_{QP}(x) - U^*(x) \right) = \mathcal{O}(||x - x_0||^2) \quad (32)$$

and the result follows by inverting the system (29) and (32). This system is indeed invertible: Due to assumption A4 it follows that $\nabla_U G_{\mathcal{A}_0}(U_0; x_0) z = 0$ for all $z \in \mathcal{F}$. Since $G_{0,\mathcal{A}_0} = \nabla_U G_{\mathcal{A}_0}(U_0; x_0)$, it is clear that $\mathcal{F} = \text{null}(G_{0,\mathcal{A}_0})$ and assumptions A2 and A3 (and in particular eq. (12)) ensures that assumption A5 holds and non-singularity of

$$\begin{pmatrix} H_0 & G_0^T \\ G_0 & 0 \end{pmatrix}$$

follows from Lemma 16.1 in [17].

□

Theorem 3 concerns only a small neighborhood of x_0 and is therefore of limited computational use. We therefore proceed by deriving some quantitative estimates and bounds on the cost and solution errors, as well as the maximum constraint violation. The solution error bound is defined as

$$\rho \triangleq \max_{x \in X_0} |w^T(\mu(0, U_{QP}(x)) - \mu(0, U^*(x)))| \quad (33)$$

where $X_0 \subset X$ is arbitrary, and w is a vector with positive weights. Likewise, we define the cost error bound

$$\varepsilon \triangleq \max_{x \in X_0} |V(U_{QP}(x); x) - V^*(x)| \quad (34)$$

where $V^*(x) \triangleq V(U^*(x); x)$. In addition, one may compute the maximum constraint violation

$$\delta \triangleq \max_{x \in X_0} \omega^T G(U_{QP}(x); x) \quad (35)$$

where ω is a vector of non-negative weights. Typically, the elements of w corresponding to the first sample of the trajectory will be positive, while the remaining will be zero since in receding horizon control the primary interest is the first sample of the trajectory. The maximum constraint violation (35) can be computed by solving an NLP, while the solution and cost error bounds (33) and (34) are not easily computed without introducing additional assumptions or allowing underestimation. A further problem is that they require computation of the exact $U^*(x)$ for several x, which relies on the solution of several NLPs and is therefore expensive. Obvious estimation techniques for ρ and ε is to take the maximum over a finite number of points X_0, such as extreme points (vertices), points generated by Monte Carlo methods, or combinations. It should be emphasized that these methods can underestimate the bounds.

5 Convexity

For the case when V and G are convex functions, it is possible to derive a guaranteed bound on ε from knowledge of $U^*(x)$ only at all the vertices $\mathcal{V} = \{v_1, v_2, ..., v_M\}$ of the bounded polyhedron X_0, similar to [11] and chapter 9.2 of [7]. Define the affine function $\overline{V}(x) \triangleq \overline{V}_0 x + \bar{l}_0$ as the solution to the following LP:

$$\min_{\overline{V}_0, \bar{l}_0} \left(\overline{V}_0 v + \bar{l}_0\right) \tag{36}$$

subject to

$$\overline{V}_0 v_i + \bar{l}_0 \geq V^*(v_i), \qquad \text{for all } i \in \{1, 2, ..., M\} \tag{37}$$

Likewise, define the convex piecewise affine function

$$\underline{V}(x) \triangleq \max_{i \in \{1,2,...,M\}} \left(V^*(v_i) + \nabla^T V^*(v_i)(x - v_i)\right) \tag{38}$$

If V^* is not differentiable at v_i, then $\nabla V^*(v_i)$ is taken as any sub-gradient of V^* at v_i:

Theorem 4 *If V and G are jointly convex (in U and x) on the bounded polyhedron X_0, then $\underline{V}(x) \leq V^*(x) \leq \overline{V}(x)$ for all $x \in X_0$.*

Proof. It is shown in [20, 7] that the joint convexity of V and G implies convexity of V^* on X_0. Let $x \in X_0$ be arbitrary, and consider the convex combination $x = \sum_i \alpha_i v_i$ where $\alpha_i \geq 0$ satisfies $\sum_i \alpha_i = 1$:

$$V^*(x) \leq \sum_{i=1}^{M} \alpha_i V^*(v_i) \leq \sum_{i=1}^{M} \alpha_i \left(\overline{V}_0 v_i + \bar{l}_0\right) = \overline{V}_0 x + \bar{l}_0$$

The lower bound $\underline{V}$ follows from the convexity of V^*, since $V^*(x) \geq V^*(v) + \nabla^T V^*(v)(x - v)$ for all $v \in X_0$ [21].
□

This immediately gives the following bounds on the cost function error $-\varepsilon_1 \leq V^*(x) - V(U_{QP}(x); x) \leq \varepsilon_2$, where

$$\varepsilon_1 = \max_{x \in X_0} \left(V(U_{QP}(x); x) - \underline{V}(x)\right) \tag{39}$$

$$\varepsilon_2 = \max_{x \in X_0} \left(\overline{V}(x) - V(U_{QP}(x); x)\right) \tag{40}$$

Hence, the cost error bound $\tilde{\varepsilon} \triangleq \max(\varepsilon_1, \varepsilon_2) \geq \varepsilon$ can be computed by solving two NLPs. It is straightforward to generalize both the upper linear and lower PWL bounds to more accurate PWL bounds by solving an NLP at one or more additional points in X_0, [7]. A solution error bound can be shown to exist as in chapter 9.7 of [7].

6 Algorithm

So far it has been established that under some regularity conditions, local mp-QP solutions give accurate approximation to the mp-NLP solution when restricted to a sufficiently small subset $X_0 \subset X$. It remains to determine a sub-partition of the polyhedral region X such that the local mp-QP solutions associated with each region are sufficiently accurate. We suggest the following algorithm to approximate the mp-NLP solution, based on recursive sub-partitioning guided by the approximation errors discussed above.

Algorithm 1 (approximate mp-NLP)

Step 1. Let $X_0 := X$.

Step 2. Select x_0 as the Chebychev center of X_0, by solving an LP.

Step 3. Compute $U_0 = U^*(x_0)$ by solving the NLP (5)-(6) with $x(0) = x_0$.

Step 4. Compute the local mp-QP problem (17) - (18) at (U_0, x_0).

Step 5. Estimate the approximation errors ε, ρ and δ on X_0.

Step 6. If $\varepsilon > \overline{\varepsilon}$, $\rho > \overline{\rho}$, or $\delta > \overline{\delta}$, then sub-partition X_0 into polyhedral regions.

Step 7. Select a new X_0 from the partition. If no further sub-partitioning is needed, go to step 8. Otherwise, repeat Steps 2-7 until the tolerances $\overline{\varepsilon}$, $\overline{\rho}$ and $\overline{\delta}$ are respected in all polyhedral regions in the partition of X.

Step 8. For all sub-partitions X_0, solve the mp-QP (17) - (18) using the mp-QP solver [3].
□

Computation of the approximation errors in Step 5 are carried out based on the results in section 5 if the cost function and constraints are known to be convex. If not, we suggest to estimate error bounds by solving NLPs at a number of points in X_0, typically the vertices and possibly other points. If the convexity assumption does not hold, this seems to be a fairly robust strategy. The sub-partitioning in Step 6 is based on a heuristic criterion where the error at the vertices are used to select one axis-orthogonal hyperplane to split X_0. The hyperplane is selected such that the error at the vertices (before splitting) across the hyperplane is maximum.

7 Example: Compressor surge control

Consider the following 2nd-order compressor model [22, 23] with x_1 being normalized mass flow, x_2 normalized pressure and u normalized mass flow through a close coupled valve in series with the compressor

$$\dot{x}_1 = B\left(\Psi_e(x_1) - x_2 - u\right) \tag{41}$$

$$\dot{x}_2 = \frac{1}{B}\left(x_1 - \Phi(x_2)\right) \tag{42}$$

The following compressor and valve characteristics are used

$$\Psi_e(x_1) = \psi_{c0} + H\left(1 + 1.5\left(\frac{x_1}{W} - 1\right) - 0.5\left(\frac{x_1}{W} - 1\right)^3\right)$$

$$\Phi(x_2) = \gamma \,\text{sign}(x_2)\sqrt{|x_2|}$$

with $\gamma = 0.5$, $B = 1$, $H = 0.18$, $\psi_{c0} = 0.3$ and $W = 0.25$. The control objective is to avoid surge, i.e. stabilize the system. This may be formulated as

$$l(x, u) = \alpha(x - x^*)^T(x - x^*) + \kappa u^2$$

$$S(x) = Rv^2 + \beta(x - x^*)^T(x - x^*)$$

with $\alpha, \beta, \kappa, \rho \geq 0$ and the setpoint $x_1^* = 0.40$, $x_2^* = 0.60$ corresponds to an unstable equilibrium point. We have chosen $\alpha = 1$, $\beta = 0$, and $\kappa = 0.08$. The horizon is chosen as $T = 12$, which is split into $N = p = 15$ equal-sized intervals, leading to a piecewise constant control input parameterizaton. Valve capacity requires the constraint $0 \leq u(t) \leq 0.3$ to hold, and the pressure constraint $x_2 \geq 0.4 - v$ avoids operation too far left of the operating point. The variable $v \geq 0$ is

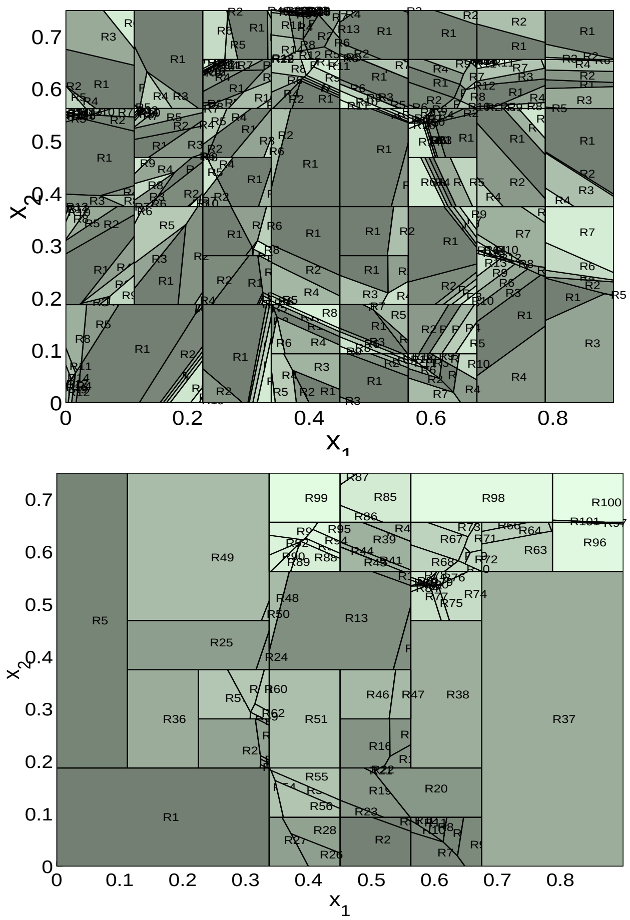

Figure 1: State space partition (top), and after reduction (bottom).

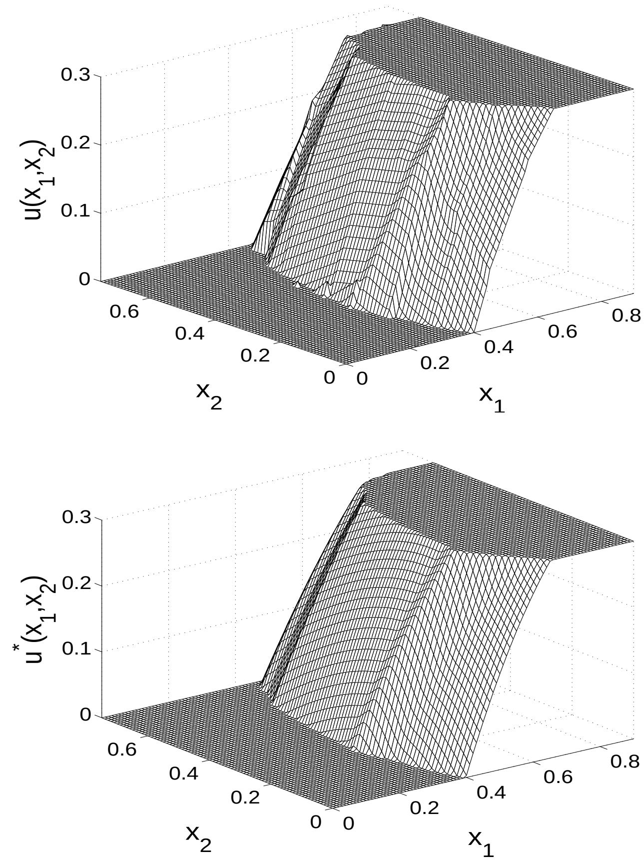

Figure 2: Piecewise linear approximate feedback control law (top) and exact feedback control law (bottom).

a slack variable introduced in order to avoid infeasibility and $R = 8$ is a large weight. Numerical analysis of the cost function shows that it is non-convex. It should be remarked that the constraints on u and v are linear, such that any mp-QP solution is feasible for the mp-NLP. The bounds ε and ρ are estimates by computing the errors at the vertices only, and the tolerances $\overline{\varepsilon} = 0.5$ and $\overline{\rho} = 0.03$ were applied. The mp-NLP contains 16 free variables, 47 constraints and 2 parameters. It is solved in 320 sec. using MATLAB with the NAG Foundation Toolbox on a 1 GHz Pentium III. The partition contains 379 regions, resulting from 45 mp-QPs, cf. Figure 1. This can be reduced to 101 polyhedral regions without loss of accuracy in a postprocessing step, where regions with the same solution at the first sample are joined whenever their union remains polyhedral, as in [4]. The computed approximate PWL feedback is shown in Figure 2, together with the exact feedback computed by solving the NLP on a dense grid. The corresponding optimal costs are shown in Figure 3, and simulation results are shown in Figure 4, where the controller is switched on after $t = 20$. We note that it quickly stabilizes the deep surge oscillations. Euler integration with step size 0.02 is applied to solve the ODE.

By generating a search tree using the method of [24], the PWL mapping with 379 regions can be represented as a binary search tree with 329 nodes, of depth 9. Real-time evaluation of the controller therefore requires 49 arithmetic operations, in the worst case, and 1367 numbers needs to be stored in real-time computer memory.

8 Conclusions

An mp-NLP algorithm has been proposed and implemented. Guaranteed properties have been established when the problem is convex, but quite often, dynamic optimization problems are not convex (or at least cannot be proven to be convex). To get a robust algorithm that may also work well when convexity is violated, the partition and termination criteria are based on combining the convexity theory with heuristics. The algorithm is shown to work satisfactory on a compressor surge control simulation example.

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References

[1] T. Gal, *Postoptimal analyses, parametric programming and related topics,* Berlin: de Gruyter, 1995.

[2] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos, “The explicit solution of model predictive control via multiparametric quadratic programming,” in *Proc. American Control Conference, Chicago*, 2000, pp. 872–876.

[3] P. Tøndel, T. A. Johansen, and A. Bemporad, “An algorithm for multi-parametric quadratic programming and explicit MPC solutions,” in *Proc. IEEE Conf. Decision and Control, Orlando*, 2001, pp. TuP11–4.

[4] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos,

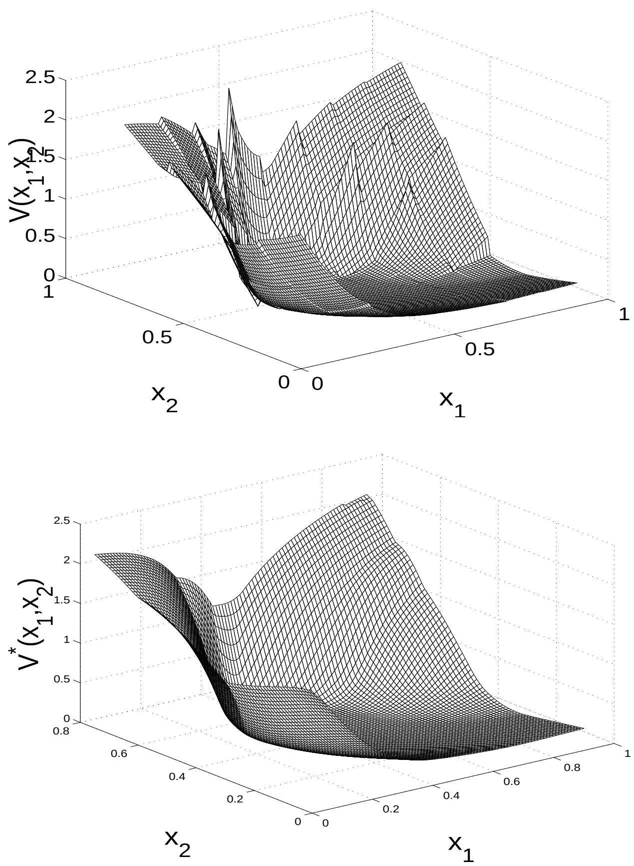

Figure 3: Optimal costs of the approximate feedback control law (top) and exact feedback control law (bottom).

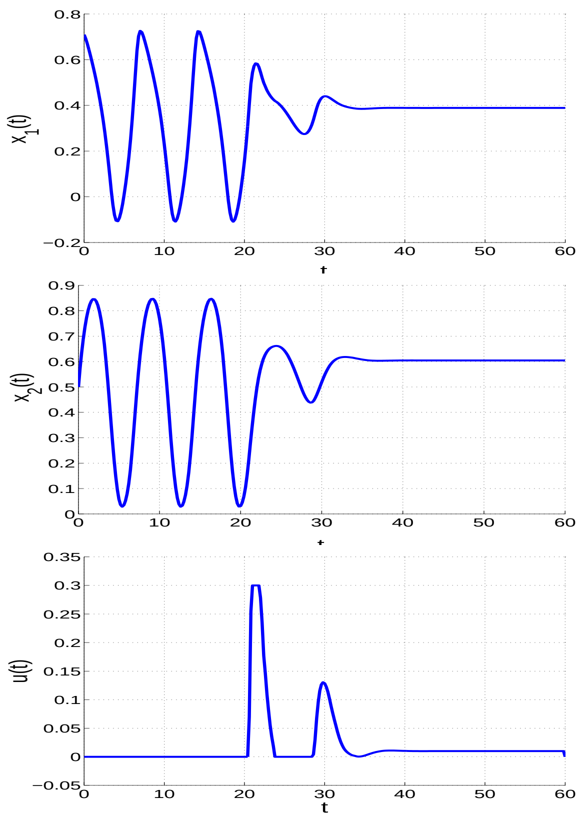

Figure 4: Simulation of compressor with approximate explicit nonlinear MPC. The solution with the exact explicit MPC cannot be distinguished graphically.

"The explicit linear quadratic regulator for constrained systems," *Automatica*, vol. 38, pp. 3–20, 2002.

[5] A. Bemporad, F. Borrelli, and M. Morari, "Optimal controllers for hybrid systems: Stability and piecewise linear explicit form," in *Proc. Conference on Decision and Control, Sydney*, 2000.

[6] A. Kojima and M. Morari, "LQ control for constrained continuous-time systems: An approach based on singular value decomposition," in *Proc. IEEE Conf. Decision and Control, Orlando*, 2001, pp. FrP08–3.

[7] A. V. Fiacco, *Introduction to sensitivity and stability analysis in nonlinear programming*, Orlando, Fl: Academic Press, 1983.

[8] E. S. Levitin, *Perturbation theory in mathematical programming*, Wiley, 1994.

[9] J. Guddat, F. Guerra Vazquez, and H. Th. Jongen, *Parametric optimization: Singularities, pathfollowing and jumps*, Wiley, 1990.

[10] A. Bemporad and C. Filippi, "Suboptimal explicit MPC via approximate quadratic programming," in *Proc. IEEE Conf. Decision and Control, Orlando*, 2001, pp. FrP08–5.

[11] T. A. Johansen and A. Grancharova, "Approximate explicit model predictive control implemented via orthogonal search tree partitioning," in *Preprints, IFAC World Congress, Barcelona*, 2002, http://www.itk.ntnu.no/ansatte/Johansen_Tor.Arne/mpqp-tree-ifac.pdf.

[12] T. A. Johansen, I. Petersen, and O. Slupphaug, "On explicit suboptimal LQR with state and input constraints," in *Proc. IEEE Conf. Decision and Control, Sydney*, 2000, pp. TuM05–6.

[13] T. Parisini and R. Zoppoli, "A receding-horizon regulator for nonlinear systems and a neural approximation," *Automatica*, vol. 31, pp. 1443–1451, 1995.

[14] T. Parisini and R. Zoppoli, "Neural approximations for multistage optimal control of nonlinear stochastic systems," *IEEE Trans. on Automatic Control*, vol. 41, pp. 889–895, 1996.

[15] D. P. Bertsekas and J. N. Tsitsiklis, *Neuro-dynamic Programming*, Athena Scientific, Belmont, 1998.

[16] D. Kraft, "On converting optimal control problems into nonlinear programming problems," in *Computational Mathematical Programming*, K. Schittkowski, Ed., vol. F15, pp. 261–280. NATO ASI Series, Springer-Verlag, 1985.

[17] J. Nocedal and S. J. Wright, *Numerical Optimization*, Springer-Verlag, New York, 1999.

[18] M. Kojima, "Strongly stable stationary solutions in nonlinear programs," in *Analysis and Computation of Fixed Points*, S. M. Robinson, Ed., pp. 93–138. Academic Press, New York, 1980.

[19] D. Ralph and S. Dempe, "Directional derivatives of the solution of a parametric nonlinear program," *Mathematical Programming*, vol. 70, pp. 159–172, 1995.

[20] O. L. Mangasarian and J. B. Rosen, "Inequalities for stochastic nonlinear programming problems," *Operations Research*, vol. 12, pp. 143–154, 1964.

[21] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, New Jersey, 1970.

[22] E. M. Greitzer, "Surge and rotating stall in axial flow compressors, part i: Theoretical compression system model," *J. Engineering for Power*, vol. 98, pp. 190–198, 1976.

[23] J. T. Gravdahl and O. Egeland, "Compressor surge control using a close-coupled valve and backstepping," in *Proc. American Control Conference, Albuquerque, NM.*, 1997, vol. 2, pp. 982 –986.

[24] P. Tøndel, T. A. Johansen, and A. Bemporad, "Evaluation of piecewise affine control via binary search tree," *Automatica*, vol. 39, pp. accepted for publication, 2003.