Exponential Stability of Regularized Moving Horizon Observer for N-Detectable Nonlinear Systems

Dan Sui and Tor Arne Johansen

***Abstract*— A moving horizon observer is analyzed for nonlinear N-detectable discrete-time systems. Conditions for global exponential stability are given. The algorithm can be implemented with regularization to ensure graceful degradation of performance when the data are not exciting. This regularization relies on monitoring an estimate of a Hessian-like matrix and conditions for local exponential convergence are given.**

I. INTRODUCTION

Moving Horizon State Estimator (MHE) makes use of a finite memory moving window of both current and recent measurement data in a least-squares criterion, possibly in addition to a state estimate and covariance matrix estimate to set the initial conditions at the beginning of the data window, see [1], [2], [3] for different formulation relying on somewhat different assumptions.

Uniform observability is typically assumed for stability or convergence proofs. However, uniform observability is a restrictive assumption that is likely not to hold in certain interesting and important state estimation applications. This is in particular true for some combined state and parameter estimation problems, for systems that are detectable but not observable, or when the data may not be persistently exciting.

Consider the following discrete-time nonlinear system:

$$x(t+1) = f(x(t),u(t)) \tag{1a}$$

$$y(t) = h(x(t),u(t)), \tag{1b}$$

where $x(t) \in R^{n_x}$, $u(t) \in R^{n_u}$ and $y(t) \in R^{n_y}$ are respectively the state, input and measurement vectors, and t is the discrete time index. In this paper a nonlinear MHE approach based on the work [4], [5], [6] is extended. In [4], [5], strongly detectable systems [7] are considered, and convergence on compact sets is analyzed. In [6] the strong detectability conditions is relaxed by using the concept of incremental input-to-state stability [8] and provide global conditions for exponential stability. The present paper provides additional results on the choice of weighting matrix in the moving horizon cost function in order to achieve regularization when data are not persistently exciting, based on monitoring of information contents using the singular value decomposition, similar to [4], [5]. Conditions for local exponential stability are derived.

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D. Sui and T. A. Johansen are with Department of Engineering Cybernetics, Norwegian University of Science and Technology, Trondheim, Norway. D. Sui is currently with SINTEF. tor.arne.johansen@itk.ntnu.no

II. PRELIMINARIES

The following notation and nomenclature is used. $||x||_A^2 = x^T Ax$ by $A \geq 0$. For two vectors $x \in R^n$ and $y \in R^m$ we let $\text{col}(x,y)$ denote the column vector in R^{n+m} where x and y are stacked into a single column. The composition of two functions f and g is written $f \circ g(x) = f(g(x))$. A function $\varphi : R^+ \to R^+$ is called a K-function if $\varphi(0) = 0$ and it is strictly increasing. A function $\varphi : R^+ \to R^+$ is called a K_∞-function if $\varphi \in K$ and it is radially unbounded. A function $\beta : R^+ \times R^+ \to R^+$ is called a KL-function if for each fixed $k \in R^+$, $\beta(\cdot,k) \in K$ and for each fixed $s \in R^+$, $\beta(s,\cdot)$ is non-increasing and $\lim_{k\to\infty} \beta(s,k) = 0$. For a sequence $\{z(j)\}$ for $j \geq 0$, $z_{[t]}$ denotes the truncation of $\{z(j)\}$ at time t, i.e. $z_{[t]} = \{z(j)\}$ for $0 \leq j \leq t$.

We define the notion of global incremental input-to-state stability [8].

Definition 1: The system (1a) is globally incrementally input-to-state stable (δISS), if there exist a KL-function θ and a K_∞-function γ_u such that for any $t \geq 0$, any initial conditions $x(0), \bar{x}(0) \in R^{n_x}$ and any $u_{[t-1]}, \bar{u}_{[t-1]}$ with $u(j), \bar{u}(j) \in R^{n_u}, 0 \leq j \leq t-1$, the following is true:

$$||x(t) - \bar{x}(t)|| \leq \theta(||x(0) - \bar{x}(0)||, t) + \gamma_u(||u_{[t-1]} - \bar{u}_{[t-1]}||). \tag{2}$$

Definition 2: (δISS-Lyapunov Function) A continuous function $V : R^{n_x} \times R^{n_x} \to R \geq 0$ is called a δISS-Lyapunov function for the system (1a) if the following holds:

1. $V(0,0) = 0$.
2. There exist K_∞-functions α_1, α_2 such that for any $x, \bar{x}$,

$$\alpha_1(||x - \bar{x}||) \leq V(x, \bar{x}) \leq \alpha_2(||x - \bar{x}||). \tag{3}$$

3. There exists a K-function σ, such that for any $x, \bar{x}$ and any couple of input signals $u, \bar{u}$

$$V(f(x,u), f(\bar{x},\bar{u})) - V(x,\bar{x}) \leq -\alpha_3(||x - \bar{x}||) + \sigma(||u - \bar{u}||) \tag{4}$$

with α_3 positive on R^+.

The following results are taken from [6]

Theorem 1: If there exists a δISS-Lyapunov function for the system (1a), then the system (1a) is δISS. Moreover, the δISS property holds with

$$\theta(s,t) = \alpha_1^{-1}(2\rho^t \alpha_2(s)), \quad \gamma_u(s) = \alpha_1^{-1}(\frac{2\sigma(s)}{1-\rho}), \tag{5}$$

for some $\rho \in [0,1)$.

Proof: Given in Appendix A for completeness. ■

Definition 3: (Global Quadratic δISS-Lyapunov Function) A continuous function $V(x, \bar{x}, P) = ||x - \bar{x}||_P^2$ with $P =$

$P^T > 0$ is called a global quadratic δISS-Lyapunov function for the system (1a) if the following holds:
1. $V(0,0,P) = 0$.
2. There exist a symmetric matrix $Q > 0$ and a symmetric matrix $\Delta > 0$, such that for any $x, \bar{x}$ and any couple of input signals $u, \bar{u}$,

$$V(f(x,u), f(\bar{x},\bar{u}), P) - V(x,\bar{x},P) \leq -V(x,\bar{x},Q) + V(u,\bar{u},\Delta). \quad (6)$$

Lemma 1: Consider the system (1a) with f globally Lipschitz and continuously differentiable. The system has a quadratic δISS-Lyapunov function V with a symmetric matrix $Q > 0$ and a symmetric matrix $\Delta > 0$ and a Lyapunov matrix $P = P^T > 0$ if for all $x, \bar{x} \in R^{n_x}$ and $u \in R^{n_u}$

$$2\Lambda^T(x,\bar{x})P\Lambda(x,\bar{x}) - P \leq -Q, \quad (7)$$

for some symmetric $Q > 0$ and $\Lambda(x,\bar{x}) = \int_0^1 \frac{\partial}{\partial x} f((1-s)x + s\bar{x}, u)ds$.

Proof: The proof is given in [6]. ■

III. Nonlinear MHE problem formulation

The $N+1$ consecutive measurements of outputs and inputs until time t are denoted as

$$Y_t = \begin{bmatrix} y(t-N) \\ y(t-N+1) \\ \vdots \\ y(t) \end{bmatrix}, \quad U_t = \begin{bmatrix} u(t-N) \\ u(t-N+1) \\ \vdots \\ u(t) \end{bmatrix}. \quad (8)$$

To express Y_t as a function of $x(t-N)$ and U_t, denote $f^{u(t)}(x(t)) = f(x(t),u(t))$ and $h^{u(t)}(x(t)) = h(x(t),u(t))$, and note from (1b) that the following algebraic map can be formulated [2]:

$$\begin{aligned} Y_t &= H(x(t-N),U_t) = H_t(x(t-N)) \\ &= \begin{bmatrix} h^{u(t-N)}(x(t-N)) \\ h^{u(t-N+1)} \circ f^{u(t-N)}(x(t-N)) \\ \vdots \\ h^{u(t)} \circ f^{u(t-1)} \circ \cdots \circ f^{u(t-N)}(x(t-N)) \end{bmatrix}. \end{aligned}$$

Definition 4: The system (1a)-(1b) is globally *N-observable* if there exists a K-function φ such that for any x_1, x_2 there exists a U_t such that

$$\varphi(||x_1 - x_2||^2) \leq ||H(x_1,U_t) - H(x_2,U_t)||^2.$$

Definition 5: The input U_t is said to be *N-exciting* for the globally N-observable system (1a)-(1b) at time t if there exists a K-function φ_t such that for any x_1, x_2 satisfying

$$\varphi_t(||x_1 - x_2||^2) \leq ||H(x_1,U_t) - H(x_2,U_t)||^2.$$

Define the N-information vector at time t as

$$I(t) = col(y(t-N),\ldots,y(t),u(t-N),\ldots,u(t)). \quad (9)$$

When a system is not N-observable, it is not possible to reconstruct exactly all the state components from the N-information vector. However, in some cases one may be able to reconstruct exactly at least some components, based on the N-information vector, and the remaining components can be reconstructed asymptotically. This corresponds to the notion of detectability [7], where we suppose there exists a coordinate transform $T : R^{n_x} \to R^{n_x}$

$$d = \begin{pmatrix} \xi \\ z \end{pmatrix} = T(x) \quad (10)$$

that lead to the following form

$$\xi(t+1) = F_1(\xi(t), z(t), u(t)) \quad (11a)$$
$$z(t+1) = F_2(z(t), u(t)) \quad (11b)$$
$$y(t) = g(z(t), u(t)). \quad (11c)$$

This transform effectively partitions the state x into an observable sub-state $z \in R^{n_z}$ and an unobservable sub-state $\xi \in R^{n_\xi}$, and the following global detectability definition can be given, [6]:

Definition 6: The system (1a)-(1b) is globally *N-detectable* if

1) There exists a coordinate transform T that brings the system in the form (11a)-(11c).
2) The sub-system (11b)-(11c) is globally N-observable.
3) The sub-system (11a) has a global quadratic δISS-Lyapunov function.

Definition 7: The input U_t is said to be *N-exciting* for a globally N-detectable system (1a)-(1b) at time t if it is N-exciting for the associated globally N-observable sub-system (11b)-(11c) at time t.
The following regularity properties are assumed throughout this paper:
(A1) The functions f and h are globally Lipschitz and twice differentiable.
(A2) The function T is continuously differentiable, globally Lipschitz and bounded away from singularity for all $x \in R^{n_x}$ such that $T^{-1}(x)$ is well defined. It is also assumed that $T^{-1}(x)$ is globally Lipschitz.
(A3) The system (1a)-(1b) is globally N-detectable and the input U_t is N-exciting for all $t \geq 0$. Moreover, the sub-system (11a) has a global quadratic δISS-Lyapunov function $V(\xi_1, \xi_2, P_\xi)$ such that $P_\xi = P_\xi^T > 0$ with symmetric matrices $Q_\xi > 0$ and $\Delta_z > 0, \Delta_u > 0$, that is,

$$\begin{aligned} &V(F_1(\xi_1,z_1,u_1), F_1(\xi_2,z_2,u_2), P_\xi) - V(\xi_1,\xi_2,P_\xi) \\ &\leq -V(\xi_1,\xi_2,Q_\xi) + V(u_1,u_2,\Delta_u) + V(z_1,z_2,\Delta_z). \end{aligned} \quad (12)$$

(A4) $x(t)$, $u(t)$ and $y(t)$ are bounded for all $t \geq 0$.
The proposed MHE problem consists in estimating, at any time $t = N, N+1, \ldots$, the state vectors $x(t-N), \ldots, x(t)$, on the basis of a priori estimates $\bar{x}(t-N)$ and the information vector $I(t)$. It is assumed that an a priori estimator is determined from the last estimate $\hat{x}^o(t-N-1|t-1)$, by

$$\bar{x}(t-N) = f(\hat{x}^o(t-N-1|t-1), u(t-N-1)).$$

A convergent estimator is pursued by minimizing the following weighted regularized least-squares criterion

$$\begin{aligned} J(\hat{x}(t-N|t); \bar{x}(t-N), I(t)) &= ||Y_t - H(\hat{x}(t-N|t),U_t)||^2_{W_t} \\ &+ ||\hat{x}(t-N|t) - \bar{x}(t-N)||^2_M \end{aligned} \quad (13)$$

with $M \geq 0$ and $W_t \geq 0$ being symmetric time-varying weight matrices. The first term is a standard least-squares term, while the second term provides a regularizing effect as it penalizes deviation from an open loop observer. The regularization leads to graceful degradation of performance if data are not N-exciting and the system is subject to uncertainty such as noise and unknown disturbances.

Let $J_t^o = \min_{\hat{x}(t-N|t)} J(\hat{x}(t-N|t); \bar{x}(t-N), I(t))$, let $\hat{x}^o(t-N|t)$ be the associated optimal estimate, and the estimation error is defined as

$$e(t-N) = x(t-N) - \hat{x}^o(t-N|t). \tag{14}$$

IV. Stability of nonlinear MHE

In the stability analysis we will need to make use of the coordinate transform into observable and unobservable states, although we emphasize that knowledge of this transform is not needed for the implementation of the observer. To express Y_t as a function of $z(t-N)$ and U_t, the following algebraic mapping can be formulated similar to the mapping H:

$$\begin{aligned} Y_t &= G(z(t-N), U_t) = G_t(z(t-N)) \\ &= \begin{bmatrix} g^{u(t-N)}(z(t-N)) \\ g^{u(t-N+1)} \circ F_2^{u(t-N)}(z(t-N)) \\ \vdots \\ g^{u(t)} \circ F_2^{u(t-1)} \circ \cdots \circ F_2^{u(t-N)}(z(t-N)) \end{bmatrix}. \end{aligned} \tag{15}$$

In order to state the stability result and the proof, the following definitions are given:

$$\begin{aligned} \hat{\Phi}_t &= \hat{\Phi}_t(z(t-N), \hat{z}^o(t-N|t)) \\ &= \int_0^1 \frac{\partial}{\partial z} G((1-s)z(t-N) + s\hat{z}^o(t-N|t), U_t) ds, \\ \hat{\Upsilon}_t &= \Upsilon_t(\check{x}(t-N-1), \hat{x}^o(t-N-1|t-1)) \\ &= \int_0^1 \frac{\partial}{\partial x} f((1-s)\check{x}(t-N-1) \\ &\quad + s\hat{x}^o(t-N-1|t-1), u(t-N-1)) ds, \\ \hat{\Gamma}_t &= \Gamma_t(\check{d}(t-N-1), \hat{d}^o(t-N-1|t-1)) \\ &= \int_0^1 \frac{\partial}{\partial d} T^{-1}((1-s)\check{d}(t-N-1) \\ &\quad + s\hat{d}^o(t-N-1|t-1)) ds, \end{aligned}$$

where $\check{x}(t-N-1) = T^{-1}(\check{d}(t-N-1))$ with $\check{d}(t-N-1) = \mathtt{col}(\hat{\xi}^o(t-N-1|t-1), z(t-N-1))$ and $\hat{d}^o(t-N-1|t-1) = T(\hat{x}^o(t-N-1|t-1))$.

Theorem 2: Suppose that assumptions (A1)-(A4) hold. Then for any $M \geq 0$, there exists a sufficiently large weight matrix $W_t \geq 0$ such that the observer error dynamics is globally exponentially stable.

Proof: The proof is found in [6], and repeated here for completeness since it is needed in the proof of the main result in the next section. The basic idea behind the proof consists in establishing upper and lower bounds on the optimal cost J_t^o, and use these bounds to show convergence.

Lower bound on the optimal cost J_t^o

Using the fact that system (1a)-(1b) can be transformed using (10), there exist $d(t-N) = T(x(t-N))$, $\hat{d}^o(t-N|t) = T(\hat{x}^o(t-N|t))$ such that in the new coordinates, the system is in the form of (11a)-(11c). Note that the first term in the right-hand side of expression (13) in the new coordinates can be rewritten as

$$\begin{aligned} &||Y_t - G(\hat{z}^o(t-N|t), U_t)||_{W_t}^2 \\ &= ||G(z(t-N), U_t) - G(\hat{z}^o(t-N|t), U_t)||_{W_t}^2. \end{aligned}$$

From Proposition 2.4.7 in [9], since (A1) and (A2) hold, we have

$$\begin{aligned} &G(z(t-N), U_t) - G(\hat{z}^o(t-N|t), U_t) \\ &= \hat{\Phi}_t(z(t-N), \hat{z}^o(t-N|t))(z(t-N) - \hat{z}^o(t-N|t)). \end{aligned}$$

Then we have

$$||Y_t - G(\hat{z}^o(t-N|t), U_t)||_{W_t}^2 = ||z(t-N) - \hat{z}^o(t-N|t)||^2_{\hat{\Phi}_t^T W_t \hat{\Phi}_t}. \tag{17}$$

Taking zero as the lower bound on the second term of (13) we get

$$J_t^o \geq ||z(t-N) - \hat{z}^o(t-N|t)||^2_{\hat{\Phi}_t^T W_t \hat{\Phi}_t}.$$

Upper bound on the optimal cost J_t^o

Let $\check{x}(t-N) = f(\check{x}(t-N-1), u(t-N-1))$. From the optimality of $\hat{x}^o(t-N|t)$, we have $J_t^o \leq J(\check{x}(t-N); \bar{x}(t-N), I(t))$. Combining the upper and lower bound on J_t^o,

$$J(\check{x}(t-N); \bar{x}(t-N), I(t)) \geq ||z(t-N) - \hat{z}^o(t-N|t)||^2_{\hat{\Phi}_t^T W_t \hat{\Phi}_t}. \tag{18}$$

Proof of the stability.

Considering the cost function $J(\check{x}(t-N); \bar{x}(t-N), I(t))$, $||Y_t - H(\check{x}(t-N), U_t)||_{W_t}^2 = ||G(z(t-N), U_t) - G(z(t-N), U_t)||_{W_t}^2 = 0$. Also, from Proposition 2.4.7 in [9],

$$\begin{aligned} &\check{x}(t-N) - \bar{x}(t-N) = \hat{\Upsilon}_t(\check{x}(t-N-1) - \hat{x}^o(t-N-1|t-1)), \\ &\check{x}(t-N-1) - \hat{x}^o(t-N-1|t-1) \\ &= \hat{\Gamma}_t(\check{d}(t-N-1) - \hat{d}^o(t-N-1|t-1)) \\ &= \hat{\Gamma}_t \begin{bmatrix} \hat{\xi}^o(t-N-1|t-1) - \hat{\xi}^o(t-N-1|t-1) \\ z(t-N-1) - \hat{z}^o(t-N-1|t-1) \end{bmatrix} \\ &= \hat{\Gamma}_t \eta^T (z(t-N-1) - \hat{z}^o(t-N-1|t-1)), \end{aligned}$$

where $\eta = [\mathbf{0}_{n_z \times n_\xi}, I_{n_z \times n_z}]$. Let $\Omega_t = \hat{\Gamma}_t \eta^T$. We have

$$\begin{aligned} &||\check{x}(t-N) - \bar{x}(t-N)||_M^2 \\ &= ||z(t-N-1) - \hat{z}^o(t-N-1|t-1)||^2_{\Omega_t^T \hat{\Upsilon}_t^T M \hat{\Upsilon}_t \Omega_t}. \end{aligned}$$

Therefore,

$$\begin{aligned} &||z(t-N) - \hat{z}^o(t-N|t)||^2_{\hat{\Phi}_t^T W_t \hat{\Phi}_t} \\ &\leq ||z(t-N-1) - \hat{z}^o(t-N-1|t-1)||^2_{\Omega_t^T \hat{\Upsilon}_t^T M \hat{\Upsilon}_t \Omega_t}. \end{aligned} \tag{19}$$

Consider a Lyapunov function

$$V(s(t)) = ||s_1(t)||_{P_1}^2 + ||s_2(t)||_{P_2}^2, \tag{20}$$

where $s(t) = \mathrm{col}(s_1(t), s_2(t))$, $P_1 > 0$ and $P_2 = P_\xi$ (P_ξ is given in (12)) for all $t \geq 0$. Let

$$s_1(t) = z(t-N) - \hat{z}^o(t-N|t), \; s_2(t) = \xi(t-N) - \hat{\xi}^o(t-N|t).$$

In the following $V(s(t))-V(s(t-1))<0, \forall s(t)\neq 0$ for some W_t is shown.

$$V(s(t))-V(s(t-1)) \\ = ||s_1(t)||^2_{P_1} - ||s_1(t-1)||^2_{P_1} + ||s_2(t)||^2_{P_2} - ||s_2(t-1)||^2_{P_2}.$$

Considering the optimization problem (13), it is easy to know that $\hat{\xi}^o(t-N|t) = \underline{\xi}(t-N)$. Since (A3) holds, then there exists a global quadratic δISS-Lyapunov function such that (12) is true. Then,

$$||s_2(t)||^2_{P_2} - ||s_2(t-1)||^2_{P_2} \leq -||s_2(t-1)||^2_{Q_\xi} + ||s_1(t-1)||^2_{\Delta_z}. \tag{21}$$

Therefore, we know that

$$V(s(t))-V(s(t-1)) \leq -||s_2(t-1)||^2_{Q_\xi} + ||s_1(t)||^2_{P_1} \\ - ||s_1(t-1)||^2_{P_1} + ||s_1(t-1)||^2_{\Delta_z}.$$

Since (A1)-(A4) hold, $\hat{\Phi}_t$ has full rank and $||\hat{\Phi}_t^T\hat{\Phi}_t|| \geq \varepsilon I$ for some $\varepsilon>0$, there always exists W_t such that

$$\hat{\Phi}_t^T W_t \hat{\Phi}_t \geq P_1. \tag{22}$$

It follows that

$$||s_1(t)||^2_{P_1} \leq ||s_1(t)||^2_{\hat{\Phi}_t^T W_t \hat{\Phi}_t} \leq ||s_1(t-1)||^2_{\Omega_t^T \hat{\Upsilon}_t^T M \hat{\Upsilon}_t \Omega_t}.$$

Then, we have

$$||s_1(t)||^2_{P_1} - ||s_1(t-1)||^2_{P_1} + ||s_1(t-1)||^2_{\Delta_z} \leq \\ ||s_1(t-1)||^2_{\Omega_t^T \hat{\Upsilon}_t^T M \hat{\Upsilon}_t \Omega_t} - ||s_1(t-1)||^2_{P_1} + ||s_1(t-1)||^2_{\Delta_z}.$$

Since $||\hat{\Upsilon}_t||$ and $||\hat{\Gamma}_t||$ are bounded, there always exists a sufficiently large weight matrix W_t such that for all $t\geq 0$

$$\hat{\Phi}_t^T W_t \hat{\Phi}_t \geq P_1, \tag{23a}$$
$$P_1 \geq \Omega_t^T \hat{\Upsilon}_t^T M \hat{\Upsilon}_t \Omega_t + \Delta_z + \tilde{\Delta}, \tag{23b}$$
$$W_t \geq 0, \tag{23c}$$

for some arbitrary symmetric $\tilde{\Delta}>0$. Then we have

$$V(s(t))-V(s(t-1)) \leq -||s_1(t-1)||^2_{\tilde{\Delta}} - ||s_2(t-1)||^2_{Q_\xi}, \tag{24}$$

which implies that $s(t)$ is globally exponentially stable. Since (A2) holds, it is easy to obtain that the error dynamics is globally exponentially stable. ∎

V. Selecting weight parameters

This section presents the main result of the paper. From (23), we know that the condition on W_t depends on $\hat{\Phi}_t, \hat{\Upsilon}_t, \hat{\Gamma}_t, M$ and Δ_z. Unfortunately, since $\hat{\Phi}_t$ depends on the unknown state we cannot monitor it. Hence, we have to rely on some approximation or estimate of $\hat{\Phi}_t$. Since (A1)-(A2) hold, from Proposition 2.4.7 in [9], we have

$$Y_t - H(\hat{x}^o(t-N|t),U_t) = \tilde{\Phi}_t(x(t-N)-\hat{x}^o(t-N|t)), \\ Y_t - G(\hat{z}^o(t-N|t),U_t) = \hat{\Phi}_t(z(t-N)-\hat{z}^o(t-N|t)),$$

where $\tilde{\Phi}_t = \tilde{\Phi}_t(x(t-N),\hat{x}^o(t-N|t)) = \int_0^1 \frac{\partial}{\partial x}H((1-s)x(t-N)+s\hat{x}^o(t-N|t),U_t)ds$. Since $Y_t - H(\hat{x}^o(t-N|t),U_t) = Y_t - G(\hat{z}^o(t-N|t),U_t)$,

$$\tilde{\Phi}_t(x(t-N)-\hat{x}^o(t-N|t)) = \hat{\Phi}_t(z(t-N)-\hat{z}^o(t-N|t)).$$

and

$$x(t-N)-\hat{x}^o(t-N|t) \\ = \Gamma_{t+1}(d(t-N),\hat{d}^o(t-N|t))(d(t-N)-\hat{d}^o(t-N|t)),$$

Let

$$\Gamma_{t+1} = \Gamma_{t+1}(d(t-N),\hat{d}^o(t-N|t)) \tag{25}$$
$$= \int_0^1 \frac{\partial}{\partial d} T^{-1}((1-s)d(t-N-1) \\ +s\hat{d}^o(t-N-1|t-1))ds. \tag{26}$$

With $z=\eta d$, we have

$$\tilde{\Phi}_t \Gamma_{t+1} = \hat{\Phi}_t \eta \Rightarrow \hat{\Phi}_t = \tilde{\Phi}_t \Gamma_{t+1} \eta^T. \tag{27}$$

Suppose that $||e(t-N)||$ is sufficiently small. Then the following approximations can be made by neglecting higher order terms

$$\tilde{\Phi}_t \approx \tilde{\Phi}_t^a = \tilde{\Phi}_t(\hat{x}^o(t-N|t),\hat{x}^o(t-N|t)) \\ = \frac{\partial H}{\partial x}(\hat{x}^o(t-N|t),U_t),$$

and $\hat{\Phi}_t \approx \hat{\Phi}_t^a = \tilde{\Phi}_t^a \Gamma_{t+1}\eta^T,\ \Gamma_t \approx \hat{\Gamma}_t$. In this paper we propose to choose the matrix M such that

$$M = \beta I_{n_x}, \tag{28}$$

where $\beta \geq 0$ is a scalar and define

$$\Sigma_t = \tilde{\Phi}_t^{a^T} W_t \tilde{\Phi}_t^a. \tag{29}$$

Since $||\Gamma_{t+1}||$ is always bounded, there always exist a positive scalar γ such that

$$\Gamma_{t+1}^T \Sigma_t \Gamma_{t+1} \geq \gamma \Sigma_t. \tag{30}$$

Similarly, since $||\hat{\Upsilon}_t||, ||\Gamma_t||$ are bounded, there always exist a positive scalar δ such that

$$\delta M \geq \Gamma_t^T \hat{\Upsilon}_t^T M \hat{\Upsilon}_t \Gamma_t. \tag{31}$$

Since (A3) holds, there always exists a non-negative scalar τ such that

$$\tau \eta \eta^T \geq \Delta_z. \tag{32}$$

Theorem 3: Suppose that assumptions (A1)-(A4) hold. For any given $\beta \geq 0$ and $\tilde{\Delta} = \upsilon \eta \eta^T > 0$ with a scalar $\upsilon > 0$, if the choice of $W_t \geq 0$ satisfies

$$\eta \Sigma_t \eta^T \geq \frac{\delta\beta + \tau + \upsilon}{\gamma} I_{n_z}, \tag{33}$$

then the observer error dynamics is locally exponentially stable.

Proof: From (23), it is easy to know that if the choice of $W_t \geq 0$ satisfies the following inequality,

$$\hat{\Phi}_t^T W_t \hat{\Phi}_t \geq \Omega_t^T \hat{\Upsilon}_t^T M \hat{\Upsilon}_t \Omega_t + \Delta_z + \tilde{\Delta}, \tag{34}$$

then the observer error dynamics is exponentially stable. Suppose $||e(t-N)||$ is sufficiently small, then by neglecting higher-order terms

$$\hat{\Phi}_t^T W_t \hat{\Phi}_t = \eta \Gamma_{t+1}^T \Sigma_t \Gamma_{t+1} \eta^T \geq \gamma \eta \Sigma_t \eta^T,$$
$$\delta\beta\eta\eta^T \geq \eta \Gamma_t^T \hat{\Upsilon}_t^T M \hat{\Upsilon}_t \Gamma_t \eta^T = \Omega_t^T \hat{\Gamma}_t^T M \hat{\Gamma}_t \Omega_t,$$
$$\tau\eta\eta^T \geq \Delta_z,$$
$$\tilde{\Delta} = \upsilon\eta\eta^T.$$

Therefore, since (33) holds for some $W_t \geq 0$, the observer error dynamics is locally exponentially stable. ∎

There are many methods to find W_t such that (33) holds. Here one possible W_t is provided, similar to [4], [5]. Consider a singular value decomposition (SVD) [10]

$$\tilde{\Phi}_t^a = \tilde{U}_t \tilde{S}_t \tilde{V}_t^T. \tag{35}$$

Any singular value (diagonal elements of the matrix $\tilde{S}_t$) that is zero or close to zero indicates that a linear combination of states is unobservable or the input is not N-exciting. Moreover, the corresponding row of the V_t matrix will indicate which linear combination of states cannot be estimated (locally). The Jacobian has the structural property that its rank will be no larger than $dim(z) = n_z$, due to a certain manifold being unobservable. The N-excitation of data may therefore be monitored through the robust computation of the rank of the Jacobian matrix using the SVD. We know that the convergence depends on W_t being chosen such that (33) holds. To pursue this objective, we propose to choose W_t such that, whenever possible,

$$W_t = \bar{W}_t^T \bar{W}_t, \tag{36}$$

with

$$\bar{W}_t = \sqrt{\alpha} \tilde{V}_t \tilde{S}_{\rho,t}^{+} \tilde{U}_t^T$$

where $\alpha > 0$ is a scalar, and the thresholded pseudo-inverse $\tilde{S}_{\rho,t}^{+} = \mathrm{diag}(0,...,0,1/\sigma_{t,1},..., 1/\sigma_{t,\ell})$ where $\sigma_{t,1},..., \sigma_{t,\ell}$ are the singular values larger than some $\rho > 0$ and the zeros correspond to small singular values whose inverse is set to zero [10]. Then we have

$$\Sigma_t = \tilde{\Phi}_t^{a^T} W_t \tilde{\Phi}_t^a = \alpha D,$$

where $D = \mathrm{diag}(0,...,0,1,...,1)$. For N-exciting input and $\rho > 0$ sufficiently small, [4], such choice of W_t also satisfies $\hat{\Phi}_t^{a^T} W_t \hat{\Phi}_t^a > 0$. The problem becomes to find a suitable α such that (33) holds. A sufficient condition to choose α is

$$\alpha \geq (\delta\beta + \tau + \upsilon)/\gamma. \tag{37}$$

It should be necessary to note that the choice of α satisfying (37) is mostly relevant as a qualitative guideline rather than as a practical tuning method, since the scalars in (37) may be both hard to compute, and will in many cases also be conservative compared to the linear matrix inequality conditions (23).

For inputs that are not N-exciting, the parameter $\rho > 0$ may be tuned in order to enhance robustness of the algorithm such that W_t gives zero weight on state combinations that are not excited by the given input [4], [5]. The effectiveness of this appraoch is studied in some case studies [11], [12].

VI. Numerical example

Consider the following system

$$\dot{x}_1 = -4x_1 + x_2 \tag{38a}$$
$$\dot{x}_2 = -x_2 + x_3 u \tag{38b}$$
$$\dot{x}_3 = 0 \tag{38c}$$
$$y = x_2 + v. \tag{38d}$$

It is clear that x_1 is not observable, but corresponds to a δISS system. It is also clear that the observability of x_3 will depend on the excitation u, while x_2 is generally observable. One may think of x_3 as a parameter representing an unknown gain on the input, where the third state equation is an augmentation for the purpose of estimating this parameter.

The same observability and detectability properties hold for the discretized system with sampling interval $t_f = 0.1$. It is easy to know that the sub-system (38a) has a quadratic δISS-Lyapunov function with $P_\xi = 1$ and $\Delta_z = \mathrm{diag}(0.02, 0.01)$.

In this simulation example we choose $x_0 = [4, -7, 2], \bar{x}_0 = [3, -5.9, -1]$. Choose $N = 2$ and $\tilde{\Delta} = I_{2\times 2}$. Measurement noise, with independent uniformly distributed $v \in [-0.5, 0.5]$, is added to the base case. The input is chosen with periods without informative data as follows: During $0 \leq t < 30t_f$, $u = 0$. During $30t_f \leq t < 60t_f$, u is discrete-time white noise. During $60t_f \leq t \leq 120t_f$, $u = 0$. In the simulation, true system has an input disturbance with $u - 0.15$, and the model used in the MHE observer (13) has an input disturbance with $u - 0.3$. In the following figures, true states are shown in solid line; estimated states of proposed work are shown in dash-dot line.

- Case 1: Choose $\beta = 0.8$, $\alpha = 1.3$ and $\delta = 0.058$. The simulation result is shown in Figure 1.
- Case 2: Choose $\beta = 0$ and $\alpha = 1$ and $\delta = 0.01$. The simulation result is shown in Figure 2.

The regularization is achieved by $\beta > 0$ since otherwise the parameter estimation will be mainly dominated by noise, as shown by case 2. The threshold $\delta > 0$ will effectively turn off the updating of the un-excited states which is seen in Case 1 for $t < 30t_f$ and $t > 60t_f$, any may prevent undesired drift of the estimates.

VII. Conclusions

We propose a regularization-based adaptive weight selection method for nonlinear moving horizon estimators, similar to [4], [5]. The class of nonlinear systems is slightly extended by considering a global N-detectability condition introduced in [6]. Conditions for exponential convergence are given, and the weight selection method is illustrated with simulations.

Appendix A–The proof of Theorem 1

The proof is similar to the one in [13]. From the hypothesis we know that $\alpha_1(||x-\bar{x}||) \leq V(x,\bar{x}) \leq \alpha_2(||x-\bar{x}||)$. For $x, \bar{x} \in R^{n_x} \setminus \{0\}$, due to $V(x,\bar{x}) \leq \alpha_2(||x-\bar{x}||)$, we obtain

$$V(x,\bar{x}) - \alpha_3(||x-\bar{x}||) \leq V(x,\bar{x}) - \alpha_3(||x-\bar{x}||)\frac{V(x,\bar{x})}{\alpha_2(||x-\bar{x}||)}$$
$$= \big(1 - \frac{\alpha_3(||x-\bar{x}||)}{\alpha_2(||x-\bar{x}||)}\big)V(x,\bar{x}).$$

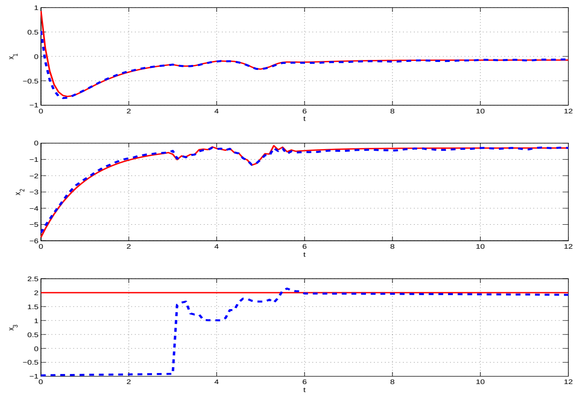

Fig. 1. Simulation results of case 1.

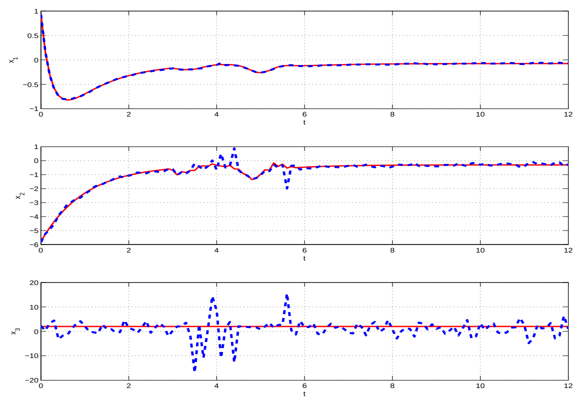

Fig. 2. Simulation results of case 2.

Let $\rho_1(\alpha_2,\alpha_3) = 1 - \frac{\alpha_3(||x-\bar{x}||)}{\alpha_2(||x-\bar{x}||)}$. Now we will show that $\rho_1(\alpha_2,\alpha_3) \in [0,1)$. Since (4) holds for $u-\bar{u}=0$,

$$\begin{aligned} 0 &\le V(f(x,u), f(\bar{x},\bar{u})) \le V(x,\bar{x}) - \alpha_3(||x-\bar{x}||) \\ &\le \alpha_2(||x-\bar{x}||) - \alpha_3(||x-\bar{x}||). \end{aligned}$$

From the above, $\rho_1(\alpha_2,\alpha_3) \in [0,1)$. Then it is easy to find $\rho \in [0,1)$ such that $\rho \ge \rho_1(\alpha_2,\alpha_3)$. Together with $V(0,0) - \alpha_3(||0||) = 0 \le \rho V(0,0)$, we have that

$$V(x(t+1),\bar{x}(t+1)) \le \rho V(x(t),\bar{x}(t)) + \sigma(||u(t)-\bar{u}(t)||).$$

We can apply the above inequality repetitively, which yields:

$$\begin{aligned} &V(x(t+1),\bar{x}(t+1)) \\ &\le \rho^{t+1} V(x(0),\bar{x}(0)) + \sum_{i=0}^{t} \rho^i \sigma(||u(t-i)-\bar{u}(t-i)||) \\ &\le \rho^{t+1} V(x(0),\bar{x}(0)) + \sigma(||u_{[t]}-\bar{u}_{[t]}||)\frac{1}{1-\rho}. \end{aligned}$$

Then we have

$$\begin{aligned} &\alpha_1(||x(t+1)-\bar{x}(t+1)||) \\ &\le \rho^{t+1}\alpha_2(||x(0)-\bar{x}(0)||) + \sigma(||u_{[t]}-\bar{u}_{[t]}||)\frac{1}{1-\rho}. \end{aligned}$$

The fact $\alpha_1 \in K_\infty$ implies $\alpha_1^{-1} \in K_\infty$. Then

$$\begin{aligned} &||x(t+1)-\bar{x}(t+1)|| \\ &\le \alpha_1^{-1}\big(\rho^{t+1}\alpha_2(||x(0)-\bar{x}(0)||) + \sigma(||u_{[t]}-\bar{u}_{[t]}||)\frac{1}{1-\rho}\big) \\ &\le \alpha_1^{-1}\big(2\max\big(\rho^{t+1}\alpha_2(||x(0)-\bar{x}(0)||), \\ &\quad \sigma(||u_{[t]}-\bar{u}_{[t]}||)\frac{1}{1-\rho}\big)\big) \\ &\le \alpha_1^{-1}\big(2\rho^{t+1}\alpha_2(||x(0)-\bar{x}(0)||)\big) \\ &\quad + \alpha_1^{-1}\big(2\sigma(||u_{[t]}-\bar{u}_{[t]}||)\frac{1}{1-\rho}\big). \end{aligned}$$

When $\rho = 0$ we have that

$$\begin{aligned} &||x(t)-\bar{x}(t)|| \le \alpha_1^{-1}\big(2\sigma(||u_{[t-1]}-\bar{u}_{[t-1]}||)\big) \\ &\le \theta(||x(0)-\bar{x}(0)||,t) + \alpha_1^{-1}\big(2\sigma(||u_{[t-1]}-\bar{u}_{[t-1]}||)\big), \end{aligned}$$

for any $\theta \in KL$. For $\rho \in (0,1)$, let $\theta(s,t) = \alpha_1^{-1}(2\rho^t\alpha_2(s))$. It is easy to know that $\theta \in KL$. Now let $\gamma_u(s) = \alpha_1^{-1}(\frac{2\sigma(s)}{1-\rho})$. Since $1/(1-\rho) > 0$, it follows that $\gamma_u \in K$. Hence the system (1a) is δISS.

REFERENCES

[1] C. V. Rao, J. B. Rawlings, and D. Q. Mayne. Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximation. *IEEE Trans. Automatic Control*, 48:246–258, 2003.

[2] P. E. Moraal and J. W. Grizzle. Observer design for nonlinear systems with discrete-time measurement. *IEEE Transactions Automatic Control*, 40:395–404, 1995.

[3] A. Alessandri, M. Baglietto, and G. Battistelli. Moving-horizon state estimation for nonlinear discrete-time systems: New stability results and approximation schemes. *Automatica*, 44:1753–1765, 2008.

[4] D. Sui and T. A. Johansen. Moving horizon observer with regularization for detectable nonlinear systems without persistence of excitation. *IFAC Symposium on Nonlinear Control, Bologna*, 2010.

[5] D. Sui and T. A. Johansen. Moving horizon observer with regularization for detectable systems without persistence of excitation. *Int. J. Control*, 84:1041–1054, 2011.

[6] D. Sui and T. A. Johansen. On the exponential stability of moving horizon observer for globally *n*-detectable nonlinear systems. *Asian J. Control, accepted for publication*, 2011.

[7] P. E. Moraal and J. W. Grizzle. Asymptotic observers for detectable and poorly observable systems. In *IEEE Conf. Decision and Control, New Orleans*, pages 109–114, 1995.

[8] D.Angeli. A Lyapunov approach to incremental stability properties. *IEEE Trans. Automatic Control*, 47:410–421, 2002.

[9] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications*. Springer-Verlag New York, 1983.

[10] G. H. Golub and C. F. van Loan. *Matrix computations*. Oxford University Press, 1983.

[11] D. Sui and T. A. Johansen. Moving horizon estimation for tire-road friction during braking. *IEEE Multi-conference of Systems and Control, Yokohama*, 2010.

[12] M. Paasche, T. A. Johansen, and L. S. Imsland. Regularized and adaptive nonlinear moving horizon estimation of bottomhole pressure during oil well drilling. *IFAC World Congress, Milano*, 2011.

[13] M. Lazar. *Model predictive control of hybrid systems: stability and robustness*. PhD thesis, Eindhoven University of Technology, 2006.